On Homogeneous Models of Fluted Languages (Extended Abstract)

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Abstract

We study the fluted fragment of first-order logic which is often viewed as a multi-variable non-guarded extension to various systems of description logics lacking role-inverses. In this paper we show that satisfiable fluted sentences (even under reasonable extensions) admit special kinds of "nice" models which we call globally/locally homogeneous. Homogeneous models allow us to simplify methods for analysing fluted logics with counting quantifiers and establish a novel result for the decidability of the (finite) satisfiability problem for the fluted fragment with periodic counting. More specifically, we will show that the (finite) satisfiability problem for the language is Tower-complete. If only two variable are used, computational complexity drops to NExpTime-completeness. We supplement our findings by showing that generalisations of fluted logics with counting, such as the adjacent fragment with counting, have finite and general satisfiability problems which are, respectively, Σ^0_1- and Π^0_1-complete. Additionally, satisfiability becomes Σ^1_1-complete if periodic counting quantifiers are permitted.

Keywords

Fluted Fragment, Fluted Fragment with Periodic Counting, Adjacent Fragment, Adjacent Fragment with Counting, Adjacent Fragment with Periodic Counting, Decidable Fragments of First-Order Logic.

1. Introduction

The *fluted fragment* (denoted $\mathcal{FL}$) is a fragment of first-order logic in which, roughly put, variables appear in predicates following the order in which they were quantified. For illustrative purposes, we translate the sentence "Every conductor nominates their favorite soloist to play at every concert" into this language as follows:

$$\forall x_1 \Big(cond(x_1) \rightarrow \exists x_2 \big(solo(x_2) \wedge fav(x_1, x_2) \wedge \forall x_3 (conc(x_3) \rightarrow nom(x_1, x_2, x_3)) \big) \Big). \quad (1)$$

Sentences axiomatising transitivity, symmetry and reflexivity are not in the fluted fragment.

The fluted fragment is a member of *argument-sequence logics* – a family of decidable (in terms of satisfiability) fragments of first-order logic which also includes the *ordered* [1, 2], *forward* [3] and *adjacent* [4] fragments. The fluted fragment in particular is decidable in terms of satisfiability even in the presence of counting quantifiers [5] or a distinguished transitive relation [6]. Surprisingly, the satisfiability problem for $\mathcal{FL}$ under a combination of the two not only retains decidability but also has the finite model property [7]. See [8] for a survey.

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In this paper we will mostly be concerned with what we call the *fluted fragment with periodic counting* (denoted $\mathcal{FLPC}$). Formally, this language is the union of sets of formulas $\mathcal{FLPC}^{[\ell]}$ defined by simultaneous induction as follows:

(i) any atom $r(x_k, \ldots, x_\ell)$, where $x_k, \ldots, x_\ell$ is a contiguous subsequence of $x_1, x_2, \ldots$ and r is a predicate of arity $\ell - k + 1$, is in $\mathcal{FLPC}^{[\ell]}$;

(ii) $\mathcal{FLPC}^{[\ell]}$ is closed under Boolean combinations;

(iii) if $\varphi \in \mathcal{FLPC}^{[\ell+1]}$, then $\exists_{[n^{+p}]} x_{\ell+1} \varphi$ is in $\mathcal{FLPC}^{[\ell]}$ for every $n, p \in \mathbb{N}$.

Semantically, $\mathfrak{A}, \bar{a} \models \exists_{[n^{+p}]} x_{\ell+1} \varphi$ if and only if $|\{b \in A \mid \mathfrak{A} \models \varphi[\bar{a}b]\}| \in \{n + ip \mid i \in \mathbb{N}\}$. We write $\mathcal{FLPC} := \bigcup_{\ell \geq 0} \mathcal{FLPC}^{[\ell]}$ for the set of all fluted formulas with periodic counting and define the ℓ-variable fragment of $\mathcal{FLPC}$ to be the set $\mathcal{FLPC}^{\ell} := \mathcal{FLPC} \cap \mathcal{FO}^{\ell}$.

We remark that periodic counting quantifiers generalise standard (threshold) counting quantifiers which have been an object of intensive study as an extension for the fluted fragment in the past few years [5, 7]. Under this new formalism, we are allowed to write formulas requesting an even number of existential witnesses. As an example, we can express sentences as “Every orchestra hires an even number of people to play first violin” in our language:

$$\forall x_1 \Big(orch(x_1) \rightarrow \exists_{[0^{+2}]} x_2 \big(pers(x_2) \wedge \exists x_3 (1st\_viol(x_3) \wedge hires\_to\_play(x_1, x_2, x_3)) \big) \Big). \quad (2)$$

The origins of *flutedness* trace back to the works of W. V. Quine [9]. It is, however, the definition given by W. C. Purdy (in [10]) that has become widespread and will be the one we use. The popularity of Purdy’s idea of flutedness is not without cause, at least when keeping the field of description logics in mind. Indeed, after a routine translation, formulas of the description logic $\mathcal{ALC}$ are contained in the two-variable sub-fragment of $\mathcal{FLPC}$. This is even the case when $\mathcal{ALC}$ is augmented with role hierarchies, nominals and/or cardinality restrictions (possibly with modulo operations). We refer the reader to [11] for more details. In terms of expressive power, $\mathcal{FLPC}$ closely parallels $\mathcal{ALCSCC}$ – a new formalism with counting constraints expressible in quantifier-free Boolean algebra with Presburger arithmetic (see [12, 13]). Thus, noting that the guarded fragment with at least three variables becomes undecidable under counting extensions [14], and that the guarded fluted fragment has “nice” model theoretic properties such as *Craig interpolation* [15], fluted languages emerge as perfect candidates for generalising description logics in a multi-variable context.

In this paper we establish that classes of models of satisfiable $\mathcal{FLPC}$-sentences always contain a “nice” structure in which elements behave (in a sense that we will make clear) *homogeneously*. Utilising this behaviour we will show that the fluted fragment extended with periodic counting quantifiers has a decidable satisfiability problem. Intriguingly, even though periodic counting quantifiers generalise standard counting quantifiers, our methodology allows us to avoid *Presburger quantification*, which was required to establish decidability of satisfiability for $\mathcal{FL}$ with standard counting [5].

To contrast our decidability results, we show that the satisfiability problems for the fluted fragment with counting extensions become undecidable when minimal syntactic relaxations are allowed. More precisely, we show that the finite satisfiability problem for the 3-variable adjacent fragment with counting is Σ^0_1-complete. Additionally, the general satisfiability problem will be shown to be Π^0_1-complete when 4 variables are used, and Σ^1_1-complete if periodic counting is allowed. Denoting the adjacent fragment as $\mathcal{AF}$, we provide a brief survey of complexity and undecidability standings in Table 1.

The work in this paper is closely related to [16] in which decidability of satisfiability is established for the two-variable fragment with periodic counting (denoted $\mathcal{FO}^2_{\text{Pres}}$) but without a sharp complexity-theoretic bound. Our homogeneity conditions, which stem from lack of inverse relations in fluted logics, allow us to establish NExpTime-completeness for both the finite and general satisfiability problems of $\mathcal{FLPC}^2$.

The full paper is published in the proceedings of the 33rd EACSL Annual Conference on Computer Science Logic (CSL 2025) [17]. The accompanying technical report with detailed proofs is available on arxiv [18].

	$\mathcal{FL}^2$	$\mathcal{FL}^\ell$	$\mathcal{AF}^3$	$\mathcal{AF}^k$
standard	NExp-c [19]	$(\ell-2)$-NExp [20]	NExp-c [4]	$(k-2)$-NExp [4]
counting	NExp-c [21]	$(\ell-1)$-NExp [5]	Σ^0_1-c/Δ^0_1 Th 11/claim	Σ^0_1-c/Π^0_1-c Th 11/15
periodic	NExp-c Th 5	$(\ell-1)$-NExp Th 9	Σ^0_1-c/Σ^0_1-h Th 11	Σ^0_1-c/Σ^1_1-c Th 11/15

Table 1
Complexity of finite (left-hand side of "/") and general (right-hand side of "/") satisfiability problems for languages (in the top row) under quantifier extensions (on the left-most column). All complexity classes are in regard to time. $\mathcal{C}$-c ($\mathcal{C}$-h) stands for complete (hard). Here, $k \geq 4$ and $\ell \geq 3$.

2. Homogeneous Models (Briefly)

We restrict attention to normal-form fluted sentences without counting quantifiers:

$$\bigwedge_{r\in R} \forall x_1 \Big(\alpha_r(x_1) \to \forall x_2\, \gamma_r(x_1,x_2)\Big) \wedge \bigwedge_{t\in T} \forall x_1 \Big(\beta_t(x_1) \to \exists x_2\, \delta_t(x_1,x_2)\Big), \tag{3}$$

where α_r, β_t are quantifier-free $\mathcal{FL}^1$-formulas and γ_r, δ_t are quantifier free $\mathcal{FL}^2$-formulas indexed by finite sets R and T. We allow equality symbols to be present in both γ_r and δ_t.

Take Σ to be any function- and constant-free signature. A fluted ℓ-type ζ over Σ is a maximal consistent set of fluted formulas formed by taking $r(x_{\ell-k+1},\dots,x_\ell)$ or $\neg r(x_{\ell-k+1},\dots,x_\ell)$ for $r \in \Sigma \cup \{=\}$ of arity $k \leq \ell$. Each ℓ-tuple $\bar{a}$ in any given Σ-structure $\mathfrak{A}$ realises a unique fluted ℓ-type (denoted $\mathsf{ftp}^{\mathfrak{A}}[\bar{a}]$) such that[1] $\pm r(x_{\ell-k+1},\dots,x_\ell) \in \mathsf{ftp}^{\mathfrak{A}}[\bar{a}]$ iff $\mathfrak{A} \models \pm r[a_{\ell-k+1},\dots,a_\ell]$.

Now, keep the Σ-structure $\mathfrak{A}$ and take ζ to be a 1-type over Σ. Writing $A_\zeta := \{a \in A \mid \mathsf{ftp}^{\mathfrak{A}}[a] = \zeta\}$ we say that ζ is *globally homogeneous* in $\mathfrak{A}$ if there is some $a \in A_\zeta$ such that, for all $b \in A_\zeta$, the following holds:

- $\mathsf{ftp}^{\mathfrak{A}}[aa] = \mathsf{ftp}^{\mathfrak{A}}[bb]$,
- $\mathsf{ftp}^{\mathfrak{A}}[ab] = \mathsf{ftp}^{\mathfrak{A}}[ba]$, and

[1] $\pm$ stands for the negation symbol $\neg$ or lack thereof. Both instances of $\pm$ are to evaluate to the same symbol.

- $\mathsf{ftp}^{\mathfrak{A}}[ac] = \mathsf{ftp}^{\mathfrak{A}}[bc]$ for each $c \in A \setminus \{a, b\}$.

Suppose now, that ζ is *not* globally homogeneous in $\mathfrak{A}$. We proceed by modifying the structure $\mathfrak{A}$ in such a way that the resulting structure $\mathfrak{A}'$ has ζ being globally homogeneous, whilst also maintaining $\mathsf{ftp}^{\mathfrak{A}'}[cd] = \mathsf{ftp}^{\mathfrak{A}}[cd]$ for all $c \in A \setminus A_\zeta$ and $d \in A$. The crux of our construction is that, when given distinct elements $a, b \in A$, there is no relation between the fluted 2-types of ab and ba. To start the construction, let $\mathfrak{A}'$ be a structure interpreting Σ over the domain A and, for each $c \in A \setminus A_\zeta$ and $d \in A$, set $\mathsf{ftp}^{\mathfrak{A}'}[cd] := \mathsf{ftp}^{\mathfrak{A}}[cd]$. Notice that, for each $a \in A$, $\mathsf{ftp}^{\mathfrak{A}'}[a] = \mathsf{ftp}^{\mathfrak{A}}[a]$ by the previous assignment. Take any $b \in A_\zeta$ and call it the *example* element. Now, set $\mathsf{ftp}^{\mathfrak{A}'}[aa] := \mathsf{ftp}^{\mathfrak{A}}[bb]$, $\mathsf{ftp}^{\mathfrak{A}'}[ab] := \mathsf{ftp}^{\mathfrak{A}}[ba]$ and $\mathsf{ftp}^{\mathfrak{A}'}[ac] := \mathsf{ftp}^{\mathfrak{A}}[bc]$ for each $a \in A_\zeta$ and $c \in A \setminus \{a, b\}$. It is easy to verify that no redefinitions take place. Moreover, $x_1 = x_2 \in \mathsf{ftp}^{\mathfrak{A}'}[cd]$ if and only if $c = d$ for each $c, d \in A$. We claim $\mathfrak{A} \models \varphi$ implies $\mathfrak{A}' \models \varphi$; it being understood that φ is a normal-form $\mathcal{FL}^2$-sentence. To see this, we need only consider elements $a \in A_\zeta$ and verify that they meet the universal and existential requirements of φ. Let $b \in A_\zeta$ be the example element picked previously. Since $\mathsf{ftp}^{\mathfrak{A}'}[a] = \mathsf{ftp}^{\mathfrak{A}}[b] = \zeta$, we have that $\mathfrak{A}' \models \beta_t[a]$ iff $\mathfrak{A} \models \beta_t[b]$ for all $t \in T$. Supposing indeed that $\mathfrak{A} \models \beta_t[b]$, let $c \in A$ be such that $\mathfrak{A} \models \delta_t[bc]$. If $c = b$, then $\mathfrak{A}' \models \delta_t[aa]$. In case $c = a$, then $\mathfrak{A}' \models \delta_t[ab]$. If neither is the case, then $\mathfrak{A}' \models \delta_t[ac]$. Either way we have that $\mathfrak{A}', a \models \beta_t(x_1) \to \exists x_2\, \delta_t(x_1, x_2)$ as required. The argument as to why $\mathfrak{A}', a \models \alpha_r(x_1) \to \forall x_2\, \gamma_r(x_1, x_2)$ is analogous.

Notice that, in the rewiring above, elements $c \notin A_\zeta$ have not been touched; i.e. $\mathsf{ftp}^{\mathfrak{A}'}[cd] := \mathsf{ftp}^{\mathfrak{A}}[cd]$ for each $d \in A$. We may run the procedure on *every* fluted 1-type thus obtaining a model in which every 1-type is globally homogeneous. We invite the reader to think of such models as structures in which elements are "stripped of their individuality". Thus, we need only be concerned with how 1- and 2-types interact with one-another.

It should be clear that restricting attention to globally homogeneous models greatly simplifies the task of determining satisfiability. As it turns out, there are appropriate generalisations of global homogeneity for multi-variable fragments of $\mathcal{FL}$; even in the presence of syntactic extensions such as periodic counting quantifiers. We refer the reader to the full article for details.

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