Guarded Fragments Meet Dynamic Logic: The Story of Regular Guards (Extended Abstract)

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1. Introduction

The *Guarded Fragment* (GF) of first-order logic (FO), introduced by Andréka et al. [1], generalizes modal and description logics (DLs) to higher-arity relational vocabularies. Over the past 25 years, GF has become the canonical first-order fragment that balances expressive power with attractive model-theoretic properties, such as the finite model property [2], preservation theorems [3], and robust decidability under various extensions involving fixed-point operators [4] or query languages [5]. Since classical (polyadic) multi-modal and description logic formulæ embed naturally into GF via standard translations, this fragment serves as a versatile logical framework central to both theoretical studies and applications in KR and databases.

However, not all widely-used families of modal and description logics (DLs) are expressible within the scope of GF, as it cannot express properties such as transitivity or equivalence of relations. Consequently, translating transitive description logics like those from the $\mathcal{S}$ family of DLs or modal logics interpreted over equivalence frames including S5, into the guarded fragment is not directly possible. To overcome this limitation, Ganzinger et al. [6] initiated the study of *semantically-constrained guards*, an extension of GF allowing certain relations—confined to guards—to be interpreted with additional semantic constraints, notably transitivity or equivalence. This direction spurred intensive research, yielding several positive results, notably the 2ExpTime-completeness of GF extended with (conjunctions of) transitive guards (consult the works of Szwast&Tendera [7], Kazakov [8] and Kieroński&Rudolph [9]), as well as the two-variable fragment of GF augmented by transitive or equivalence closures of binary guards, established by Michaliszyn and his co-authors [10, 11]. Check Tendera's survey [12] for a comprehensive overview. On the negative side, natural extensions of GF *with equality* ($\mathrm{GF}_{\approx}$), intended to capture popular description logics from the $\mathcal{SR}$ family, turned out to be undecidable. Examples include $\mathrm{GF}_{\approx}$ with exponentiation (regular expressions that are compositions of the same letter) [13] or associative compositional axioms [8]. The decidability status of these logics without equality $\approx$ is still open. Consequently, there is no known decidable extension of GF with semantically-constrained guards captures propositional dynamic logic (PDL) and its generalizations such as the $\mathcal{Z}$ family [14] of DLs, PDL with intersection and converse (ICPDL) [15], or its higher-arity extensions of DLs such as $\mathcal{DLR}$ [16].

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2. Our Contributions

We introduce and study a novel logic called RGF, which extends (equality- and constant-free) GF by allowing ICPDL-programs as guards (*cf.* Def. 1). Our main result establishes that $\mathsf{Sat}(\mathrm{RGF})$ is 2ExpTime-complete, matching the complexity of plain GF and ICPDL. This also lifts decidability to several logics where it was previously known only in their two-variable case. Our proof employs a *fusion* technique, reducing $\mathsf{Sat}(\mathrm{RGF})$ to instances of the satisfiability problem in plain GF or two-variable RGF, which is in turn solvable via an encoding to ICPDL.

Theorem 1. *The satisfiability problem for* RGF *is 2ExpTime-complete.* ◀

We further address two questions: (i) Is the query entailment problem decidable for RGF? (ii) Is there an expressive fragment of RGF of complexity lower than 2ExpTime? We answer question (i) negatively, showing undecidability of conjunctive query entailment even for two-variable fluted GF with a single transitive guard, substantially strengthening prior results of Gottlob et al. [17, Thm. 1] on entailment of unions of conjunctive queries over GF^2 with transitive guards in three ways: our logic is more restricted (belongs to the so-called fluted fragment), our queries do not use disjunction (we use conjunctive queries rather than the unions thereof), and our proof is also applicable to the finite-model scenario (which remained open).

Theorem 2. *Both finite and general CQ entailment problems are undecidable for* RGF*, already for its fluted two-variable fragment with a single transitive guard.* ◀

By fluted formulæ we mean index-normal formulae (on any branch of its syntax tree, the i-th quantifier bounds precisely x_i) where any atom $\alpha(\bar{x})$ in the scope of a quantifier bounding x_n (but not x_{n+1}), the sequence $\bar{x}$ is a suffix of the sequence $x_1, x_2, \ldots, x_n$. Our undecidability transfers to $\mathcal{ALC}$ extended with unqualified existential restrictions with intersection of the form $\exists(\mathrm{r} \cap \mathrm{s}).\top$, inclusion axioms of the form $\mathrm{r} \subseteq \mathrm{s} \cup \mathrm{t}$, and a single transitivity statement.

For (ii) we conduct a thorough case analysis, pinpointing when subfragments of RGF admit lower complexity than 2ExpTime. We conclude that a novel forward variant of GF extended with transitive closure is the largest (in a natural sense) ExpSpace-complete fragment of RGF.

Theorem 3 (Simplified statement)**.** *The satisfiability problem for the forward subfragment* RGF *with the set of operators restricted to* $\{\cdot^{+}, \cdot^{*}, ?\}$ *is ExpSpace-complete. The inclusion of other operators makes already the fluted two-variable fragment of* RGF *2ExpTime-hard.* ◀

3. Our Motivations

We explain our motivations behind the study of the GF with regular guards in the form of a Q&A.

☛ Why the guarded fragment (GF)?

Because GF is *the canonical* extension of modal and description logics to the setting of higher-arity relations [18], heavily investigated in the last 25 years. GF is not only well-behaved both computationally [2] and model-theoretically [19], but is also robust under extensions like fixed points [4] or semantically-constrained guards [12]. It was studied also in the setting of knowledge representation in multiple recent papers [20, 21, 22, 23, 24].

☛ Do we generalize any previously studied logics?
Yes, many of them. First, as GF encodes (via the standard translation, see *e.g.* Section 2.6.1 of Baader's textbook 2017) multi-modal and description logics [18], our logic also encodes (via an analogous translation) ICPDL and its subfragments such as $\mathcal{ALC}_{\mathsf{reg}}$ or $\mathcal{SRI}$ (Horrocks et al. 2006). There also exists a natural translation from (counting-free fragment of) $\mathcal{DLR}_{\mathsf{reg}}$ (Calvanese et al. 2008) to RGF. Second, there is a long tradition of studying GF extended with semantically-constrained guards [12], *i.e.* distinguished relations (available only as guards) interpreted as *transitive* (Ganzinger et al. 1999) or equivalence [27] relations, or as transitive [10] or equivalence (Kieroński et al. 2017) closures of another relation (that may also appear only as a guard). As one can simulate transitive or equivalence relation R with S^{+} and $(\mathrm{S} \cup \mathrm{S}^{-})^{*}$ for a fresh relation S, our logic strictly extends all of the mentioned logics. Moreover, the mentioned papers concerning transitive and equivalence closures only focused on the extensions of GF^{2} (two-variable GF), and hence our logic lifts them (without $\approx$) to the case of full GF and provides the tight complexity bound. Other ideas concern GF with exponentiation (regular expressions that are composition of the same letter) [13] or associative compositional axioms [8] (*i.e.* axioms $\mathrm{R} \circ \mathrm{S} \subseteq \mathrm{T}$ where $\mathrm{R}, \mathrm{S}, \mathrm{T}$ occurring only in guards). Both of them can be easily simulated in our framework. Finally, GF with conjunctions of transitive relations in guards [8] can be expressed in RGF by employing $\cap$ operator. All of this makes RGF a desirable object of study.
☛ Are there any closely related but incomparable logics?
The closest logic is the Unary Negation Fragment [28] with regular-path expressions [29] $\mathrm{UN}_{\mathsf{reg}}$, together with its very recent generalizations with transitive closure operators [30] and guarded negation [31]. All of these logics share 2ExpTime complexity of their (formula) satisfiability problem, but their expressive powers are incomparable. Indeed, RGF is not able to express conjunctive queries, while the other logics cannot express that R^{*}-reachable elements are B-connected. Yet another related logic is $\mathrm{GNFP}^{\mathsf{up}}$ by Benedikt et al. 2016, which extends the guarded (negation) fragment [33] with fixed-point operators with unguarded parameters. The syntax of $\mathrm{GNFP}^{\mathsf{up}}$ is complicated, but the logic seems to embed ICPDL. Unfortunately, according to our understanding, such an encoding leads to a non-constant "pdepth" of the resulting formulæ, leading to a non-elementary fragment of the logic. The expressive powers of $\mathrm{GNFP}^{\mathsf{up}}$ and RGF are again incomparable and the separating examples are as before.

4. Our Logic

We work with structures over a fixed countably-infinite *equality-* and *constant-free* relational signature $\Sigma := \Sigma_{\mathrm{FO}} \uplus \Sigma_{\mathfrak{R}}$, where all predicates in $\Sigma_{\mathfrak{R}}$, called *regular predicates*, are binary. By mutual induction, we define both RGF-programs and RGF-formulæ.

Definition 1 (RGF). RGF*-programs are given by the grammar:*

$$\pi, \rho \ ::= \ \mathrm{B} \mid \bar{\mathrm{B}} \mid \pi{\circ}\rho \mid \pi{\cup}\rho \mid \pi{\cap}\rho \mid \pi^{*} \mid \pi^{+} \mid \varphi?,$$

where $\mathrm{B} \in \Sigma_{\mathfrak{R}}$ *and* φ *is an* RGF*-formula with a sole free variable. An* RGF-guard ϑ *for a formula* φ *is either an atom over* Σ_{FO} *or* $\pi(xy)$ *for some* RGF*-program* π*, such that free variables of* ϑ *include all free variables of* φ*. The set* RGF *of* RGF-formulæ *is defined with the grammar:*

$$\varphi, \varphi' \;::=\; \mathrm{A}(\bar{x}) \mid \neg\varphi \mid \varphi \land \varphi' \mid \exists x\, \varphi(x) \mid \exists \bar{x}(\vartheta \land \varphi),$$

where $\mathrm{A} \in \Sigma_{\mathrm{FO}}$ *and* ϑ *is an* RGF*-guard for* φ*. The semantics of* RGF*-programs is:*

Name	Syntax of π	Semantics $\pi^{\mathfrak{A}}$ of π in a structure $\mathfrak{A}$
Test / Predicate	$\varphi?$ / B	$\{(\mathrm{a},\mathrm{a}) \mid \mathfrak{A} \models \varphi[\mathrm{a}]\}$ / Binary relation
Converse operator	$\bar{\pi}$	$\{(\mathrm{b},\mathrm{a}) \mid (\mathrm{a},\mathrm{b}) \in \pi^{\mathfrak{A}}\}$
Concatenation	$\pi \circ \rho$	$\{(\mathrm{a},\mathrm{c}) \mid \exists \mathrm{b}.(\mathrm{a},\mathrm{b}) \in \pi^{\mathfrak{A}} \land (\mathrm{b},\mathrm{c}) \in \rho^{\mathfrak{A}}\}$
Union / Intersection	$\pi \cup \rho$ / $\pi \cap \rho$	$\pi^{\mathfrak{A}} \cup \rho^{\mathfrak{A}}$ / $\pi^{\mathfrak{A}} \cap \rho^{\mathfrak{A}}$
Kleene star/plus	π^* / π^+	$\bigcup_{i=0}^{\infty}(\pi^i)^{\mathfrak{A}}$ / $\bigcup_{i=1}^{\infty}(\pi^i)^{\mathfrak{A}}$, where $\pi^0 := \top?$ and $\pi^{i+1} := (\pi^i) \circ \pi$.

◀

Our logic generalizes a plethora of extensions of GF with semantically-constrained guards (consult the introduction). For instance, transitive and equivalence relations in guards can be simulated in RGF using R^+ and $(\mathrm{R} \cup \bar{\mathrm{R}})^*$ for a fresh binary relation R. Hence, GF+TG, the extension of GF with transitive guards, is a fragment of RGF. We explain our design choices.

☛ Why is the signature separated, *i.e.* $\Sigma := \Sigma_{\mathrm{FO}} \uplus \Sigma_{\mathcal{R}}$?
To ensure that binary predicates from $\Sigma_{\mathcal{R}}$ appear *only in guards*; otherwise, even the two-variable guarded fragment with transitivity is undecidable [6, Th. 2].
☛ Why is the equality symbol $\approx$ excluded from Σ?
Its inclusion makes our logic undecidable, already for GF^2 with compositional axioms [8, Th. 5.3.1], conjunctions of transitive guards [8, Th. 5.3.2], or exponentiation [13, Th. 3.1].
☛ Why are constant symbols excluded from Σ?
They are expected to preserve decidability, especially given that the key theorem of 2ExpTime-completeness of ICPDL [34, Th. 3.28] extends to Hybrid ICPDL.

5. Future Work

Future work may proceed along two directions. One promising path is to extend the ICPDL-based guards to more expressive formalisms, such as linear Datalog or non-binary transitive closure operators. This would help eliminate the asymmetry between the binary nature of regular guards and the higher-arity relations allowed in GF. The alternative path is to tackle the finite satisfiability problem for RGF. This problem is very challenging—even for minimal fragments of ICPDL like LoopPDL—and has resisted resolution for over 40 years, indicating that significant breakthroughs and new techniques will be required. A more pragmatic direction is to study fragments of RGF with the FMP (the finite model property). While formulæ like $\forall x_1 \exists x_2 \mathrm{R}(x_1 x_2)$ combined with either $\neg\exists x_1 \mathrm{R}^+(x_1 x_1)$ or $\forall x_1 x_2(\mathrm{R}^+(x_1 x_1) \rightarrow \mathrm{B}(x_1 x_2) \land \neg\mathrm{B}(x_2 x_1))$ destroy the FMP by enforcing infinite, non-loopable R-chains, one might hope that fluted RGF retains it. Unfortunately, with a similar counter-examples, we show:

Lemma 1. *For the set of allowed operators* Op *being either* $\{\cdot^+, ?\}$, $\{\cdot^+, \cdot^*\}$, $\{\cdot^+, \bar{\cdot}\}$, *or* $\{\cdot^+, \circ\}$ *we have that the fluted* RGF *with the operators in programs restricted to these from* Op *does not have the finite model property.* ◀

We can already prove that $\mathrm{FRGF}[\cdot^+]$ has the FMP and we are currently trying to extend our approach to $\mathrm{FRGF}[\cdot^+, \cap, \cup]$. More details are coming soon!

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