Concrete Domains Meet Expressive Cardinality Restrictions in Description Logics (Extended Abstract)

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Standard *description logics* (DLs) can encode quantitative aspects of an application domain through either *(qualified) number restrictions* or *concrete domain restrictions.* Using qualified number restrictions, we can constrain the number of role successors belonging to a certain concept by a fixed natural number: for example, $\mathsf{Human} \sqcap (\geq 3\,\mathsf{child}.\mathsf{Human}) \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}$ says that a tax break is available if one has at least three children. On the other hand, concrete domain restrictions are suitable to represent a different type of quantitative information, where concrete objects such as numbers or strings can be assigned to individuals using partial functions (*features*). For example, a tax break might only be available if the annual salary is not too high. The CI $\mathsf{Human} \sqcap (\geq 3\,\mathsf{child}.\mathsf{Human}) \sqcap \exists \mathsf{salary}.{<_{100{,}000}} \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}$ specifies at least three children and an annual salary of less than 100,000 € as eligibility criteria for a tax break.

The complexity of the concept satisfiability problem for the DL $\mathcal{ALCQ}$, which extends $\mathcal{ALC}$ with qualified number restrictions, is the same as that of $\mathcal{ALC}$, i.e. PSpace-complete without a TBox and ExpTime-complete w.r.t. an ontology comprising a TBox and an ABox [1, 2]. This result holds for both unary and binary coding of the natural numbers occurring in number restrictions. Later, it was showed that the unrestricted use of transitive roles within number restrictions can cause undecidability in the presence of role inclusion axioms [3]. In [4], it was shown that reasoning in $\mathcal{ALCSCC}$, which extends $\mathcal{ALCQ}$ with very expressive counting constraints on role successors expressed in the logic QFBAPA [5], still has the same complexity as in $\mathcal{ALC}$ and $\mathcal{ALCQ}$. In this DL, we can describe humans that have exactly as many cars as children with the concept $\mathsf{Human} \sqcap \mathsf{succ}(|\mathsf{own} \cap \mathsf{Car}| = |\mathsf{child} \cap \mathsf{Human}|)$, without having to specify the exact numbers of cars and children. Unlike $\mathcal{ALCQ}$ concepts, which form a fragment of first-order logic (FOL), the concept in the previous sentence cannot be expressed in FOL [6].

Concept satisfiability for $\mathcal{ALC}(\mathfrak{D})$, the extension of $\mathcal{ALC}$ with restrictions over a concrete domain $\mathfrak{D}$, is decidable without a TBox if $\mathfrak{D}$ is *admissible* [7], essentially requiring that satisfiability of conjunctions of constraints over the predicates of $\mathfrak{D}$ is decidable. The presence of a

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TBox leads to undecidability of $\mathcal{ALC}(\mathfrak{D})$ even for rather simple instances of $\mathfrak{D}$ [8, 9]. Decidability of concept satisfiability w.r.t. a TBox is regained by considering so-called *ω-admissible* concrete domains [10], which additionally satisfy the so-called *patchwork* and *homomorphism ω-compactness* properties. Moreover, reasoning in $\mathcal{ALC}(\mathfrak{D})$ remains in ExpTime if additionally the constraint satisfaction problem of $\mathfrak{D}$ is assumed to be in ExpTime [11]. Two examples of such concrete domains are Allen's interval algebra [12] and RCC8 [13]. Using well-known notions and results from model theory, additional ω-admissible concrete domains were exhibited in [9, 14], for example the rational numbers with comparisons $\mathfrak{Q} := (\mathbb{Q}, <, =, >)$. Decidability results for $\mathcal{ALC}(\mathfrak{D})$ in the presence of CIs for concrete domains $\mathfrak{D}$ that are not ω-admissible can be found in [15, 16, 17]. A simpler, but considerably more restrictive way of achieving decidability is to use unary concrete domains [18].

In [19], we study $\mathcal{ALCOSCC}(\mathfrak{D})$, a combination of the DLs $\mathcal{ALCSCC}$ and $\mathcal{ALC}(\mathfrak{D})$ with ω-admissible concrete domains $\mathfrak{D}$ as well as nominals ($\mathcal{O}$). Rather than only looking at the plain combination of these logics, we consider stronger forms of interaction between the numerical constraints of $\mathcal{ALCSCC}$ and the feature values over $\mathfrak{D}$. For *numerical* concrete domains, such as $\mathfrak{Q}$, we consider *mixed numerical constraints* that allow for the usage of concrete features directly in the QFBAPA constraints, e.g. to describe people that own more books than their age. We show, however, that this unrestricted combination easily leads to undecidability. For arbitrary ω-admissible concrete domains, we introduce *feature roles*, defined in terms of the relations holding among the feature values of two individuals, which can then be employed within QFBAPA constraints. An example is given by the feature role $(\mathsf{salary} < \texttt{next}\,\mathsf{salary})$, which connects an individual to all individuals that have a higher salary. One can use this to describe all persons that have a lower salary than at least half of their children with $\mathsf{succ}(|\mathsf{child} \cap (\mathsf{salary} < \texttt{next}\,\mathsf{salary})| > |\mathsf{child} \cap (\mathsf{salary} \geq \texttt{next}\,\mathsf{salary})|)$. Due to the semantics of $\mathcal{ALCSCC}$, feature roles may only connect an individual to individuals that are also role successors. We also analyze an unrestricted variant where feature roles may connect arbitrary individuals; formally, this is obtained by taking the DL $\mathcal{ALCSCC}^{++}$, where constraints range over all individuals in an interpretation [20]. Lastly, we study the extension $\mathcal{SSCC}$ of $\mathcal{ALCSCC}$ by transitive roles.

Consistency of $\mathcal{ALCOSCC}(\mathfrak{D})$. By considering ExpTime-*ω-admissible concrete domains*, which are ω-admissible concrete domains $\mathfrak{D}$ whose constraint satisfaction problem is decidable in exponential time, we derive the following decidability result.

Theorem 1. *Let $\mathfrak{D}$ be an* ExpTime*-ω-admissible concrete domain. Then consistency checking in $\mathcal{ALCOSCC}(\mathfrak{D})$ is* ExpTime*-complete.*

The ExpTime-hardness of this decision problem is trivially derived from the fact that $\mathcal{ALCOSCC}(\mathfrak{D})$ subsumes $\mathcal{ALCSCC}$. The upper bound, on the other hand, is obtained by means of an algorithm based on type elimination. In this case, we use *augmented types* that, in addition to the standard notion of type, encode information about the concrete domain restrictions and number restrictions that must be satisfied by an individual described by a type. There are few results in the literature that determine the exact complexity of reasoning in DLs with concrete domains [21, 16, 17, 11]. Only [21] and [11] consider ω-admissible concrete domains, and the ExpTime-completeness result in the former is restricted to a specific temporal concrete domain. Theorem 1 extends the results of the latter from $\mathcal{ALC}(\mathfrak{D})$ to $\mathcal{ALCOSCC}(\mathfrak{D})$ and is generic since it holds for all ExpTime-ω-admissible concrete domains.

Undecidable extensions. While Theorem 1 provides a positive result, every other extension described earlier leads to undecidability, under the natural assumption that the concrete domain $\mathfrak{D}$ is *jointly diagonal* (JD), i.e. that equality between elements of $\mathfrak{D}$ can be expressed using the relations of $\mathfrak{D}$. In the case of $\mathcal{SSCC}$, we show that undecidability holds even if we apply all the restrictions added in [3, 22] to regain decidability.

Theorem 2. *Consistency in $\mathcal{SSCC}$ is undecidable, even if numerical constraints contain no transitive roles and no constants other than 0 or 1.*

Let $\mathfrak{D}$ be a jointly diagonal and infinite concrete domain. Then the consistency problem for $\mathcal{ALCSCC}^{++}(\mathfrak{D})$ TBoxes is undecidable. If $\mathfrak{D}$ is numerical, then consistency of $\mathcal{ALCOSCC}(\mathfrak{D})$ TBoxes with mixed numerical constraints is undecidable.

The undecidability result for $\mathcal{SSCC}$ is obtained by a reduction from the tiling problem, similar to the one proposed in [3] for $\mathcal{SHN}$. For $\mathcal{ALCSCC}^{++}(\mathfrak{D})$, we show that we can reduce Hilbert's tenth problem to consistency, by adapting the reduction used in [20] used to show that concept satisfiability in $\mathcal{ALCSCC}^{++}$ with inverse roles is undecidable. Finally, the presence of mixed numerical constraints leads to undecidability, since we can encode the consistency problem for $\mathcal{ALC}(\mathfrak{D})$ with a numerical concrete domain $\mathfrak{D}$ over the natural numbers with the binary successor relation, which is known to be undecidable [14].

Reasoning with constants. Using nominals, we can internalize concept and role assertions within concept inclusions. In a similar way, we can encode *predicate assertions* of the form $P(f_1(a_1), \ldots, f_k(a_k))$ where P is a relation of the concrete domain $\mathfrak{D}$, each a_i is an individual and each f_i is a feature name. An example of a predicate assertion is $\mathsf{salary}(\texttt{Sam}) < \mathsf{salary}(\texttt{Jane})$, which intuitively asserts that Jane's salary is higher than Sam's. On the other hand, we may want to also use *feature assertions* of the form $f(a, c)$ to state that the f-value of the individual a is equal to the element c of the concrete domain $\mathfrak{D}$. With feature assertions, we can give specific values and state, for instance, that Sam's salary is 100,001 € with $\mathsf{salary}(\texttt{Sam}, 100{,}001)$. For ω-admissible domains $\mathfrak{D}$, supporting feature assertions is equivalent to support *additional singleton predicates* $=_c$ that are not part of $\mathfrak{D}$, but can be used in concepts with a similar semantics, provided that $\mathfrak{D}$ satisfies additional conditions. These conditions concern the usage of constants in constraint systems, in relation to their encoding, and are satisfied by the main known examples of ExpTime-ω-admissible concrete domains ($\mathfrak{Q}$, Allen's relations, and RCC8) under the reasonable assumptions that all numbers are given as integer fractions in binary encoding and the constants in RCC8 refer to polygonal regions in the rational plane [23, 24]. We call the resulting structures ExpTime-*ω-admissible concrete domains with constants.* We obtain the following result for *homogeneous* concrete domains $\mathfrak{D}$, i.e. where every isomorphism between finite substructures of $\mathfrak{D}$ can be extended to an isomorphism from $\mathfrak{D}$ to itself [14].

Theorem 3. *If $\mathfrak{D}$ is an* ExpTime-*ω-admissible and homogeneous concrete domain with constants, then consistency in $\mathcal{ALCOSCC}(\mathfrak{D})$ with feature assertions and additional singleton predicates is* ExpTime-*complete.*

The conference paper [19] and the technical report [25] contain a detailed discussion of all the results above. While feature roles can already express a restricted form of inverse roles, in the future, we would like to investigate the decidability and complexity of $\mathcal{ALCOISCC}(\mathfrak{D})$ with full inverse roles, which increase the complexity of classical DLs with nominals and number restrictions to NExpTime [2]. Another avenue of research is to implement a reasoner for $\mathcal{ALCOSCC}(\mathfrak{D})$, based on a suitable tableaux algorithm [10] that needs to integrate a QFBAPA solver and a concrete domain reasoner. Currently, reasoners for DLs with non-trivial concrete domains only exist for $\mathcal{ALC}(\mathfrak{D})$ and $\mathcal{EL}(\mathfrak{D})$ with so-called p-admissible concrete domains and without feature paths [26].

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